SECOND ORDER ESTIMATES AND REGULARITY FOR FULLY NONLINEAR ELLIPTIC EQUATIONS ON RIEMANNIAN MANIFOLDS

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Abstract. We derive *a priori* second order estimates for solutions of a class of fully nonlinear elliptic equations on Riemannian manifolds under some very general structure conditions. We treat both equations on closed manifolds, and the Dirichlet problem on manifolds with boundary without any geometric restrictions to the boundary except being smooth and compact. As applications of these estimates we obtain results on regularity and existence.

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1. Introduction

This is one of several papers in which we seek methods to derive *a priori* estimates for fully nonlinear elliptic equations on real or complex manifolds. Our techniques work for various classes of equations under conditions which are near optimal in many situations. In this paper we shall focus on the second order estimates for the Hessian type equations on Riemannian manifolds.

Let (M^n, g) be a compact Riemannian manifold of dimension $n \geq 2$ with smooth boundary ∂M, and $\bar{M} := M \cup \partial M$. Let f be a smooth symmetric function of n variables and χ a smooth $(0,2)$ tensor on $\bar{M}$. We consider fully nonlinear equations of the form

$$f(\lambda[\nabla^2 u + \chi]) = \psi \text{ in } M \tag{1.1}$$

where $\nabla^2 u$ denotes the Hessian of $u \in C^2(M)$ and $\lambda[\nabla^2 u + \chi] = (\lambda_1, \cdots, \lambda_n)$ are the eigenvalues of $\nabla^2 u + \chi$ with respect to the metric g.

Fully nonlinear equations of form (1.1) in $\mathbb{R}^n$ was first considered by Caffarelli, Nirenberg and Spruck in their seminal paper [5]. Following [5] we assume f is defined

in a symmetric open and convex cone $\Gamma \subset \mathbb{R}^n$ with vertex at the origin and boundary $\partial\Gamma \neq \emptyset$,

$$\Gamma^+ \equiv \{\lambda \in \mathbb{R}^n : \text{each component } \lambda_i > 0\} \subseteq \Gamma, \tag{1.2}$$

and to satisfy the standard structure conditions:

$$f_i = f_{\lambda_i} \equiv \frac{\partial f}{\partial \lambda_i} > 0 \quad \text{in } \Gamma, \quad 1 \leq i \leq n, \tag{1.3}$$

$$f \text{ is a concave function,} \tag{1.4}$$

$$\delta_{\psi,f} \equiv \inf \psi - \sup_{\partial\Gamma} f > 0; \quad \text{where } \sup_{\partial\Gamma} f \equiv \sup_{\lambda_0 \in \partial\Gamma} \limsup_{\lambda \to \lambda_0} f(\lambda). \tag{1.5}$$

According to [5] condition (1.3) ensures that equation (1.1) is elliptic for solutions $u \in C^2(M)$ with $\lambda[\nabla^2 u + \chi] \in \Gamma$; we shall call such functions *admissible*, while condition (1.4) implies the function F defined by $F(A) = f(\lambda[A])$ to be concave for $A \in \mathcal{S}^{n\times n}$ with $\lambda[A] \in \Gamma$, where $\mathcal{S}^{n\times n}$ is the set of n by n symmetric matrices. By condition (1.5), equation (1.1) *becomes* uniformly elliptic once *a priori* C^2 bounds are established for admissible solutions so that one can apply the classical Evans-Krylov theorem to obtain $C^{2,\alpha}$ estimates. So these conditions are basically indispensable to the study of equation (1.1).

The most typical equations of form (1.1) are given by $f = \sigma_k^{\frac{1}{k}}$ and $f = (\sigma_k/\sigma_l)^{\frac{1}{k-l}}$, $1 \leq l < k \leq n$ defined on the cone

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0 \text{ for } 1 \leq j \leq k\},$$

where σ_k is the k-th elementary symmetric function

$$\sigma_k(\lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}, \quad 1 \leq k \leq n.$$

These functions satisfy (1.3)-(1.4) and have other properties which have been widely used in study of the corresponding equations; see e.g. [5], [43], [49], [45], [55], [10].

The Dirichlet problem for equation (1.1) in $\mathbb{R}^n$ was extensively studied by Caffarelli, Nirenberg and Spruck [5], Ivochkina [37], Krylov [39], Wang [55], Trudinger [50], Trudinger and Wang [51], Chou and Wang [10], and the author [15], [19], among many others. In this paper we deal with equation (1.1) on general Riemannian manifolds.

Equation (1.1) was first studied by Y.-Y. Li [43] on closed Riemannian manifolds, followed by the work of Urbas [52].

A central issue in solving equation (1.1) is to derive C^2 estimates for admissible solutions, in view of the Evans-Krylov theorem. We shall be mainly concerned with estimates for second derivatives. Such estimates was first derived by Y.-Y. Li [43] for equation (1.1) with $\chi = g$ on closed manifolds of nonnegative sectional curvature. Urbas [52] was able to remove the nonnegative curvature assumption. In deriving the estimates, the presence of curvature creates terms which are difficult to control. As a result, in addition to (1.3)-(1.5) both papers needed extra assumptions which excluded the case $f = (\sigma_k/\sigma_l)^{1/(k-l)}$; see Section 5 for more discussions about the results of [43] and [52].

In order to state our main results, which cover the case $f = (\sigma_k/\sigma_l)^{1/(k-l)}$, we first introduce some notation.

For $\sigma > \sup_{\partial\Gamma} f$, define $\Gamma^\sigma = \{\lambda \in \Gamma : f(\lambda) > \sigma\}$, and we shall only consider the case $\Gamma^\sigma \neq \emptyset$. Let $\mathcal{C}_\sigma$ denote the tangent cone at infinity to the level surface $\partial\Gamma^\sigma$ which is smooth and convex by conditions (1.3) and (1.4). Let $\mathcal{C}_\sigma^+$ be the open component of $\Gamma \setminus (\mathcal{C}_\sigma \cap \Gamma)$ containing Γ^σ.

Our first main result is the following global second order estimates.

Theorem 1.1. *Let $\psi \in C^2(M \times \mathbb{R}) \cap C^1(\bar{M} \times \mathbb{R})$ and $u \in C^4(M) \cap C^2(\bar{M})$ be an admissible solution of (1.1). Suppose $a \leq u \leq b$ on $\bar{M}$ and let*

$$\underline{\psi}(x) = \min_{a \leq z \leq b} \psi(x, z), \quad \hat{\psi}(x) = \max_{a \leq z \leq b} \psi(x, z), \quad x \in \bar{M}.$$

In addition to (1.3)-(1.4), assume

$$\delta_{\underline{\psi}, f} = \inf_{\bar{M}} \underline{\psi} - \sup_{\partial\Gamma} f > 0. \tag{1.6}$$

and that there exists a function $\underline{u} \in C^2(\bar{M})$ satisfying

$$\lambda[\nabla^2 \underline{u} + \chi](x) \in \mathcal{C}^+_{\hat{\psi}(x)}, \quad \forall\, x \in \bar{M}. \tag{1.7}$$

Then

$$\max_{\bar{M}} |\nabla^2 u| \leq C_1\big(1 + \max_{\partial M} |\nabla^2 u|\big). \tag{1.8}$$

In particular, if M is closed ($\partial M = \emptyset$) then

$$|\nabla^2 u| \leq C_2 e^{C_3(u - \inf_M u)} \quad \text{on } M \tag{1.9}$$

where C_1, C_2 depend on $|u|_{C^1(M)}$ but not on $1/\delta_{\underline{\psi}, f}$ and C_3 is a uniform constant (independent of u).

As we shall see in Section 5, condition (1.7) is implied by the assumptions in [43]. By approximation we obtain the following regularity result from Theorem 1.1.

Theorem 1.2. *Let (M^n, g) be a closed Riemannian manifold and $\psi \in C^{1,1}(M \times \mathbb{R})$. Under conditions (1.3)-(1.4),* (1.5) *and* (1.7)*, any admissible weak solution (in the viscosity sense) $u \in C^{0,1}(M)$ of* (1.1) *belongs to $C^{1,1}(M)$ and* (1.9) *holds.*

By the Evans-Krylov theorem, $u \in C^{2,\alpha}(M)$, $0 < \alpha < 1$; higher regularities follow from the classical Schauder elliptic theory. In particular, $u \in C^\infty(M)$ if $\psi \in C^\infty(M)$.

Remark 1.3. Condition (1.7) is always satisfied if there is a strictly convex function on M ($\partial M \neq \emptyset$), or if $\chi \in \mathcal{C}_\sigma^+$ (for instance, if $\chi = ag$, $a > 0$ and the vertex of $\mathcal{C}_\sigma$ is the origin) for all σ. For $f = \sigma_k^{1/k}$ ($k \geq 2$), $\Gamma_n^+ \subset \mathcal{C}_\sigma^+$ for any $\sigma > 0$. See also Lemma 5.1.

Corollary 1.4. *Let (M, g) be a closed Riemannian manifold and $\psi \in C^{1,1}(M)$. In addition to (1.3)-(1.5), suppose $\chi \in \mathcal{C}_\sigma^+$ for all $\sup_{\partial\Gamma} f < \sigma \leq \sup_M \psi$. Then any admissible weak solution $u \in C^{0,1}(M)$ of* (1.1) *belongs to $C^{2,\alpha}(M)$, $0 < \alpha < 1$, and* (1.9) *holds.*

We now turn to the second order boundary estimates. We wish to derive such estimates without imposing any geometric conditions on ∂M except being smooth and compact. For simplicity we only consider the case $\psi = \psi(x)$.

Theorem 1.5. *Let $\psi \in C^1(\bar{M})$, $\varphi \in C^4(\partial M)$ and $u \in C^3(M) \cap C^1(\bar{M})$ be an admissible solution of* (1.1) *with $u = \varphi$ on ∂M. Assume f satisfies (1.3)-(1.5) and*

$$\sum f_i \lambda_i \geq 0 \quad \text{in } \Gamma. \tag{1.10}$$

Suppose that there exists an admissible subsolution $\underline{u} \in C^0(\bar{M})$ in the viscosity sense:

$$\begin{cases} f(\lambda[\underline{u}_{ij} + a_{ij}]) \geq \psi & \text{in } \bar{M}, \\ \underline{u} = \varphi & \text{on } \partial M \end{cases} \tag{1.11}$$

and that $\underline{u}$ is C^2 and satisfies

$$\lambda[\nabla^2 \underline{u} + \chi](x) \in \mathcal{C}^+_{\psi(x)} \tag{1.12}$$

in a neighborhood of ∂M. Then there exists $C_4 > 0$ depending on $|u|_{C^1(\bar{M})}$ and $1/\delta_{\underline{\psi}, f}$ such that

$$\max_{\partial M} |\nabla^2 u| \leq C_4. \tag{1.13}$$

Remark 1.6. An admissible subsolution $\underline{u} \in C^2(\bar{M})$ will automatically satisfy (1.7) provided that

$$\partial\Gamma^\sigma \cap \mathcal{C}_\sigma = \emptyset, \ \ \forall \, \sigma \in \big[\inf_M \psi, \sup_M \psi\big]. \tag{1.14}$$

Condition (1.14) excludes the linear function $f = \sigma_1$ which corresponds to the Poisson equation, but is clearly satisfied by a wide class of concave functions including $f = \sigma_k^{1/k}$, $k \geq 2$ and $f = (\sigma_k/\sigma_l)^{1/(k-l)}$ for all $1 \leq l < k \leq n$. Note that condition (1.14) holds if $\partial\Gamma^\sigma$ is strictly convex.

Applying Theorems 1.1 and 1.5 we can prove the following existence result by the standard continuity method.

Theorem 1.7. *Let $\psi \in C^\infty(\bar{M})$, $\varphi \in C^\infty(\partial M)$. Suppose f satisfies (1.3)-(1.5),* (1.10) *and that there exists an admissible subsolution $\underline{u} \in C^2(\bar{M})$ satisfying* (1.11) *and* (1.12) *for all $x \in \bar{M}$. Then there exists an admissible solution $u \in C^\infty(\bar{M})$ of the Dirichlet problem for equation* (1.1) *with boundary condition $u = \varphi$ on ∂M, provided that* **(i)** $\Gamma = \Gamma_n^+$, *or* **(ii)** *the sectional curvature of (M, g) is nonnegative, or* **(iii)** *f satisfies*

$$f_j \geq \delta_0 \sum f_i(\lambda) \ \text{if} \ \lambda_j < 0, \ \text{on} \ \partial\Gamma^\sigma \ \forall \, \sigma > \sup\nolimits_{\partial\Gamma} f. \tag{1.15}$$

When M is a smooth bounded domain in $\mathbb{R}^n$, Theorem 1.7 **(ii)** extends the previous results of Caffarelli, Nirenberg and Spruck [5], Trudinger [50] and the author [15]; see [19] for more detailed discussions. The assumptions **(i)**-**(iii)** are only needed to derive gradient estimates; see Proposition 5.3. It would be desirable to remove these assumptions.

Corollary 1.8. *Let $f = \sigma_k^{1/k}$, $k \geq 2$ or $f = (\sigma_k/\sigma_l)^{\frac{1}{k-l}}$, $0 \leq l < k \leq n$. Given $\psi \in C^\infty(\bar{M})$, $\psi > 0$ and $\varphi \in C^\infty(\partial M)$, suppose that there exists an admissible subsolution $\underline{u} \in C^2(\bar{M})$ satisfying* (1.11). *Then there exists an admissible solution $u \in C^\infty(\bar{M})$ of equation* (1.1) *with $u = \varphi$ on ∂M.*

In Theorem 1.7 there are no geometric restrictions to ∂M being made. This gives Theorem 1.7 the advantage of flexibility in applications. In general, the Dirichlet problem is not always solvable in arbitrary domains without the subsolution assumption, as in the case of Monge-Ampère equations. In the classical theory of elliptic equations, a standard technique is to use the distance function to the boundary to construct local barriers for boundary estimates. So one usually need require the

boundary to possess certain geometric properties; see e.g. [47] for the prescribed mean curvature equation and [4], [3] for Monge-Ampère equations; see also [14] and [5]. Technically, we use $u - \underline{u}$ to replace the boundary distance function in deriving the second order boundary estimates. This idea was first used by Haffman, Rosenberg and Spruck [35] and further developed in [23], [21], [16], [17] to treat the real and complex Monge-Ampère equations in general domains as well as in [15], [18] for more general fully nonlinear equations. Their results and techniques have found useful applications in some important problems; see e.g. the work of P.-F. Guan [27], [28] and papers of Chen [9], Blocki [2], and Phong and Sturm [46] on the Donaldson conjectures [11] in Kähler geometry. In [23], [24], [25] we used the techniques to study Plateau type problems for locally convex hypersurfaces of constant curvature in $\mathbb{R}^{n+1}$.

We shall also make use of $u-\underline{u}$ in the proof of the global estimate (1.8). This is one of the key ideas in this paper; see the proof in Section 3. Note that in Theorem 1.1 the function $\underline{u}$ is not necessarily a subsolution. On a closed manifold, an admissible subsolution for $\psi = \psi(x)$ must be a solution if there is a solution at all, and any two admissible solutions differ at most by a constant. This is a consequence of the concavity condition (1.4) and the maximum principle.

Similar equations where χ depends on u or ∇u (or both) also occur naturally and have received extensive study in classical differential geometry; see e.g. [20], [29], and in conformal geometry in which there is a huge literature; see for instance [6], [7], [8], [12], [30], [31], [32], [33], [34], [40], [41], [42], [48], [53], [54] and references therein. In the current paper we confine our discussion to the case $\chi = \chi(x)$, $x \in \bar{M}$.

In Section 2 we discuss some consequences of the concavity condition. Our proof of the estimates heavily depends on results in Section 2. The global and boundary estimates are derived in Sections 3 and 4, respectively. In Section 5 we briefly discuss the results of Li [43] and Urbas [52], followed by gradient estimates. We end the paper with a new example which was first brought to our attention by Xinan Ma to whom we wish to express our gratitude.

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2. The concavity condition

Let $\sigma > \sup_{\partial\Gamma} f$ and assume $\Gamma^\sigma := \{f > \sigma\} \neq \emptyset$. Then $\partial\Gamma^\sigma$ is a smooth convex noncompact complete hypersurface contained in Γ. Clearly $\Gamma^\sigma \neq \mathcal{C}_\sigma^+$ unless $\partial\Gamma^\sigma$ is a plane.

Let $\mu, \lambda \in \partial\Gamma^\sigma$. By the convexity of $\partial\Gamma^\sigma$, the open segment

$$(\mu, \lambda) \equiv \{t\mu + (1-t)\lambda : 0 < t < 1\}$$

is either completely contained in or does not intersect with $\partial\Gamma^\sigma$. Therefore,

$$f(t\mu + (1-t)\lambda) - \sigma > 0, \ \ \forall\, 0 < t < 1$$

by condition (1.3), unless $(\mu, \lambda) \subset \partial\Gamma^\sigma$.

For $R > |\mu|$, let

$$\Theta_R(\mu) \equiv \inf_{\lambda \in \partial B_R(0) \cap \partial\Gamma^\sigma} \max_{0 \leq t \leq 1} f(t\mu + (1-t)\lambda) - \sigma \geq 0.$$

Note that $\Theta_R(\mu) = 0$ if and only if $(\mu, \lambda) \subset \partial\Gamma^\sigma$ for some $\lambda \in \partial B_R(0) \cap \partial\Gamma^\sigma$, since the set $\partial B_R(0) \cap \partial\Gamma^\sigma$ is compact.

Lemma 2.1. *For $\mu \in \partial\Gamma^\sigma$, $\Theta_R(\mu)$ is nondecreasing in R. Moreover, if $\Theta_{R_0}(\mu) > 0$ for some $R_0 \geq |\mu|$ then $\Theta_{R'} > \Theta_R$ for all $R' > R \geq R_0$.*

Proof. Write $\Theta_R = \Theta_R(\mu)$ when there is no possible confusion. Suppose $\Theta_{R_0}(\mu) > 0$ for some $R_0 \geq |\mu|$. Let $R' > R \geq R_0$ and assume $\lambda_{R'} \in \partial B_{R'}(0) \cap \partial\Gamma^\sigma$ such that

$$\Theta_{R'} = \max_{0 \leq t \leq 1} f(t\mu + (1-t)\lambda_{R'}) - \sigma.$$

Let P be the (two dimensional) plane through $\mu, \lambda_{R'}$ and the origin of $\mathbb{R}^n$. There is a point $\lambda_R \in \partial B_R(0)$ which lies between μ and $\lambda_{R'}$ on the curve $P \cap \partial\Gamma^\sigma$. Note that μ, λ_R and λ_R' are not on a straight line, for (μ, λ_R) can not be part of $(\mu, \lambda_{R'})$ since $\Theta_{R_0} > 0$ and $\partial\Gamma^\sigma$ is convex. We see that

$$\max_{0 \leq t \leq 1} f(t\mu + (1-t)\lambda_R) - \sigma < \Theta_{R'}$$

by condition (1.3). This proves $\Theta_R < \Theta_{R'}$. $\square$

Corollary 2.2. *Let $\mu \in \partial\Gamma^\sigma$. The following are equivalent:*

(a) $\mu \in \mathcal{C}_\sigma$;
(b) $\Theta_R(\mu) = 0$ *for all* $R > |\mu|$;
(c) $\partial\Gamma^\sigma \cap \mathcal{C}_\sigma$ *contains a ray through* μ;

(d) *$T_\mu\partial\Gamma^\sigma \cap \mathcal{C}_\sigma$ contains a ray through μ, where $T_\mu\partial\Gamma^\sigma$ is the tangent (supporting) plane of $\partial\Gamma^\sigma$ at μ.*

Lemma 2.3. *Let $\mu \in \overline{\Gamma^\sigma}$, $\mu \notin \mathcal{C}_\sigma$. There exist positive constants ω_μ, N_μ such that for any $\lambda \in \partial\Gamma^\sigma$, when $|\lambda| \geq N_\mu$,*

$$\sum f_i(\lambda)(\mu_i - \lambda_i) \geq \omega_\mu. \tag{2.1}$$

Proof. By the concavity of f,

$$\sum f_i(\lambda)(\mu_i - \lambda_i) \geq f(\mu) - f(\lambda).$$

We see (2.1) holds if $f(\mu) > \sigma$. So we assume $\mu \in \partial\Gamma^\sigma$. By Corollary 2.2, $\Theta_R(\mu) > 0$ for all R sufficiently large, and therefore, again by the concavity of f,

$$\sum f_i(\lambda)(\mu_i - \lambda_i) \geq \max_{0\leq t\leq 1} f(t\mu + (1-t)\lambda) - \sigma \geq \Theta_R(\mu) > 0$$

for any $\lambda \in \partial B_R(0) \cap \partial\Gamma^\sigma$. Since $\Theta_R(\mu)$ is increasing in R, Lemma 2.3 holds. □

Our main results of this paper is based on the following observation.

Theorem 2.4. *Let $\mu \in \mathcal{C}_\sigma^+$. For any $0 < \varepsilon <$ dist$(\mu, \mathcal{C}_\sigma)$ there exist positive constants θ_μ, R_μ such that for any $\lambda \in \partial\Gamma^\sigma$, when $|\lambda| \geq R_\mu$,*

$$\sum f_i(\lambda)(\mu_i - \lambda_i) \geq \theta_\mu + \varepsilon \sum f_i(\lambda). \tag{2.2}$$

Proof. Since $\mu \in \mathcal{C}_\sigma^+$ and $\varepsilon < \operatorname{dist}(\mu, \mathcal{C}_\sigma)$, we see that $\mu^\varepsilon \equiv \mu - \varepsilon\mathbf{1} \in \mathcal{C}_\sigma^+$ where $\mathbf{1} = (1, \dots, 1)$. Let $\mathcal{C}(\mu^\varepsilon)$ be the tangent cone to Γ^σ with vertex μ^ε. Then $\partial\Gamma^\sigma \cap \mathcal{C}(\mu^\varepsilon)$ is compact and therefore contained in a ball $B_{R_0}(0)$ for some $R_0 > 0$. Let $\partial\Gamma_{\sigma,\mu^\varepsilon}$ denote the compact subset of $\partial\Gamma^\sigma$ bounded by $\partial\Gamma^\sigma \cap \mathcal{C}(\mu^\varepsilon)$.

Let $R > R_0$ and $\lambda \in \partial B_R(0) \cap \partial\Gamma^\sigma$. The segment $[\mu^\varepsilon, \lambda]$ goes through $\partial\Gamma_{\sigma,\mu^\varepsilon}$ at a point λ^ε. Since $f(\lambda) = f(\lambda^\varepsilon) = \sigma$, by the concavity of f we obtain

$$\sum f_i(\lambda)((\mu_i - \varepsilon) - \lambda_i) \geq \sum f_i(\lambda)(\lambda_i^\varepsilon - \lambda_i) \geq \omega_{\lambda^\varepsilon} \geq \inf_{\eta\in\partial\Gamma_{\sigma,\mu^\varepsilon}} \omega_\eta \equiv \theta_\mu > 0$$

when $R \geq R_\mu \equiv \sup_{\eta\in\partial\Gamma_{\sigma,\mu^\varepsilon}} N_\eta$. □

Theorem 2.4 can not be used directly in the proofs of (1.8) and (1.13) in the next two sections. So we modify it as follows.

Let $\mathcal{A}$ be the set of n by n symmetric matrices $A = \{A_{ij}\}$ with eigenvalues $\lambda[A] \in \Gamma$. Define the function F on $\mathcal{A}$ by

$$F(A) \equiv f(\lambda[A]).$$

Throughout this paper we shall use the notation

$$F^{ij}(A) = \frac{\partial F}{\partial A_{ij}}(A), \quad F^{ij,kl}(A) = \frac{\partial^2 F}{\partial A_{ij}\partial A_{kl}}(A).$$

The matrix $\{F^{ij}\}$ has eigenvalues $f_1, \ldots, f_n$ and is positive definite by assumption (1.3), while (1.4) implies that F is a concave function of A_{ij} [5]. Moreover, when A is diagonal so is $\{F^{ij}(A)\}$, and the following identities hold

$$F^{ij}(A)A_{ij} = \sum f_i\lambda_i,$$

$$F^{ij}(A)A_{ik}A_{kj} = \sum f_i\lambda_i^2.$$

Theorem 2.5. *Let $A \in \mathcal{A}$, $\lambda(A) \in \mathcal{C}_\sigma^+$. Then for any $0 < \varepsilon < dist(\lambda(A), \mathcal{C}_\sigma)$ there exist positive constants θ_A, R_A such that for any $B \in \mathcal{A}$ with $\lambda(B) \in \partial\Gamma^\sigma$, when $|\lambda(B)| \geq R_A$,*

$$F^{ij}(B)(A_{ij} - B_{ij}) \geq \theta_A + \varepsilon \sum F^{ii}(B). \tag{2.3}$$

Proof. Suppose first that $\lambda(A) \in \Gamma^\sigma$. Then, since $\lambda(A) \notin \mathcal{C}_\sigma$,

$$(A, B) \equiv \{tA + (1-t)B : 0 < t < 1\}$$

is completely contained in Γ^σ for any $B \in \mathcal{A}$ with $\lambda(B) \in \partial B_R(0) \cap \partial\Gamma^\sigma$ when R is sufficiently large. Therefore,

$$\Theta_R(A) \equiv \inf_{\lambda(B)\in\partial B_R(0)\cap\partial\Gamma^\sigma} \max_{0\leq t\leq 1} F(tA + (1-t)B) - \sigma > 0$$

and $\Theta_R(A)$ is increasing in R. By the concavity of F we have

$$F^{ij}(B)(A_{ij} - B_{ij}) \geq \max_{0\leq t\leq 1} F(tA + (1-t)B) - \sigma \geq \Theta_R(A)$$

In the general case, let $A^\varepsilon = A - \varepsilon I \in \mathcal{A}$ so $\lambda(A^\varepsilon) = \lambda(A) - \varepsilon\mathbf{1}$. When R is sufficiently large, for any $B \in \mathcal{A}$ with $\lambda(B) \in \partial B_R(0) \cap \partial\Gamma^\sigma$ we can find $C \in (A, B)$ such that $\lambda(C)$ is contained in the compact set $\partial\Gamma_{\sigma,\lambda(A^\varepsilon)}$. As before,

$$F^{ij}(B)(A_{ij} - \varepsilon\delta_{ij} - B_{ij}) \geq F^{ij}(B)(C_{ij} - B_{ij}) \geq \Theta_R(C).$$

This completes the proof of Theorem 2.5 in view of the compactness of $\partial\Gamma_{\sigma,\lambda(A^\varepsilon)}$. $\square$

The following inequality is taken from [26] with minor modifications. We shall need it in the boundary estimates in Section 4.

Proposition 2.6. *Let $A = \{A_{ij}\} \in \mathcal{A}$ and set $F^{ij} = F^{ij}(A)$. There is $c_0 > 0$ and an index r such that*

$$\sum_{l<n} F^{ij} A_{il} A_{lj} \geq c_0 \sum_{i \neq r} f_i \lambda_i^2. \tag{2.4}$$

Proof. Let $B = \{b_{ij}\}$ be an orthogonal matrix that simultaneously diagonalizes $\{F^{ij}\}$ and $\{A_{ij}\}$:

$$F^{ij} b_{li} b_{kj} = f_k \delta_{kl}, \quad A_{ij} b_{li} b_{kj} = \lambda_k \delta_{kl}.$$

Then

$$\sum_{l<n} F^{ij} A_{il} A_{lj} = \sum_{l<n} f_i \lambda_i^2 b_{li}^2. \tag{2.5}$$

Suppose for some i, say $i = 1$ and $0 < \theta < 1$ to be determined that

$$\sum_{l<n} b_{l1}^2 < \theta^2.$$

Then

$$b_{n1}^2 = 1 - \sum_{l<n} b_{l1}^2 > 1 - \theta^2 > 0.$$

Expanding $\det B$ by cofactors along the first column gives

$$1 = \det B = b_{11} C^{11} + \ldots + b_{(n-1)1} C^{1n-1} + b_{n1} \det D \leq c_1 \theta + |b_{n1} \det D|,$$

where C^{1j} are the cofactors and D is the $n-1$ by $n-1$ matrix

$$D = \begin{bmatrix} b_{12} & \ldots & b_{(n-1)2} \\ \vdots & \ddots & \vdots \\ b_{1n} & \ldots & b_{(n-1)n} \end{bmatrix}. \tag{2.6}$$

Therefore,

$$|\det D| \geq \frac{1 - c_1 \theta}{|b_{n1}|} \geq 1 - c_1 \theta.$$

Now expanding $\det D$ by cofactors along row $i \geq 2$ gives

$$|\det D| \leq c_2 \Big(\sum_{l<n} b_{li}^2 \Big)^{\frac{1}{2}}$$

by Schwarz inequality. Hence

$$\sum_{l<n} b_{li}^2 \geq \Big(\frac{1 - c_1 \theta}{c_2} \Big)^2. \tag{2.7}$$

Choosing $\theta < \frac{1}{2c_1}$, (2.7) and (2.5) imply

$$\sum_{l<n} F^{ij} A_{il} A_{lj} \geq c_0 \sum_{i\neq 1} f_i \lambda_i^2.$$

This proves (2.4). □

Lemma 2.7. *Suppose f satisfies (1.3), (1.4) and (1.10). Then*

$$\sum_{i\neq r} f_i \lambda_i^2 \geq \frac{1}{n} \sum f_i \lambda_i^2 \quad \text{if } \lambda_r < 0. \tag{2.8}$$

Proof. Suppose $\lambda_1 \geq \cdots \geq \lambda_n$ and $\lambda_r < 0$. By the concavity condition (1.4) we have $f_n \geq f_i > 0$ for all i and in particular $f_n \lambda_n^2 \geq f_r \lambda_r^2$. By (1.10),

$$\sum_{i\neq n} f_i \lambda_i \geq -f_n \lambda_n = f_n |\lambda_n|.$$

By Schwarz inequality,

$$f_n^2 \lambda_n^2 \leq \sum_{i\neq n} f_i \sum_{i\neq n} f_i \lambda_i^2 \leq (n-1) f_n \sum_{i\neq n} f_i \lambda_i^2.$$

Therefore,

$$\sum_{i\neq r} f_i \lambda_i^2 \geq \sum_{i\neq n} f_i \lambda_i^2 \geq \frac{1}{n} \sum_{i\neq n} f_i \lambda_i^2 + \frac{1}{n} f_n \lambda_n^2 = \frac{1}{n} \sum f_i \lambda_i^2$$

completing the proof. □

Corollary 2.8. *Suppose f satisfies (1.3)-(1.4). Then for any index r*

$$\sum f_i |\lambda_i| \leq \epsilon \sum_{i\neq r} f_i \lambda_i^2 + C\Big(1 + \frac{1}{\epsilon} \sum f_i\Big). \tag{2.9}$$

Proof. By the concavity of f,

$$f(\mathbf{1}) - f(\lambda) \leq \sum f_i (1 - \lambda_i).$$

Therefore, if $\lambda_r \geq 0$ then

$$f_r \lambda_r \leq f(\lambda) - f(\mathbf{1}) + \sum f_i + \sum_{\lambda_i<0} f_i |\lambda_i| \leq \epsilon \sum_{\lambda_i<0} f_i \lambda_i^2 + \frac{C}{\epsilon} \sum f_i + C.$$

Suppose $\lambda_r < 0$. By Lemma 2.7 we have

$$\sum f_i |\lambda_i| \leq \frac{\epsilon}{n} \sum f_i \lambda_i^2 + \frac{n}{4\epsilon} \sum f_i \leq \epsilon \sum_{i\neq r} f_i \lambda_i^2 + \frac{C}{\epsilon} \sum f_i.$$

This proves (2.9). □

3. Global bounds for the second derivatives

The goal of this section is to prove (1.8) under the hypotheses (1.3), (1.4), (1.6) and (1.7). We start with a brief explanation of our notation and basic formulas needed. Throughout the paper ∇ denotes the Levi-Civita connection of (M^n, g). The curvature tensor is defined by

$$R(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z.$$

Let $e_1, \ldots, e_n$ be local frames on M^n and denote $g_{ij} = g(e_i, e_j)$, $\{g^{ij}\} = \{g_{ij}\}^{-1}$, and $\nabla_i = \nabla_{e_i}$, $\nabla_{ij} = \nabla_i \nabla_j - \nabla_{\nabla_i e_j}$, etc. Define R_{ijkl}, R^i_{jkl} and Γ^k_{ij} respectively by

$$R_{ijkl} = \langle R(e_k, e_l)e_j, e_i \rangle, \quad R^i_{jkl} = g^{im} R_{mjkl}, \quad \nabla_i e_j = \Gamma^k_{ij} e_k.$$

For a differentiable function v defined on M^n, we identify ∇v with the gradient of v, and $\nabla^2 v$ denotes the Hessian of v which is given by $\nabla_{ij} v = \nabla_i(\nabla_j v) - \Gamma^k_{ij} \nabla_k v$. Recall that $\nabla_{ij} v = \nabla_{ji} v$ and

$$\nabla_{ijk} v - \nabla_{jik} v = R^l_{kij} \nabla_l v, \tag{3.1}$$

$$\nabla_{ijkl} v - \nabla_{ikjl} v = R^m_{ljk} \nabla_{im} v + \nabla_i R^m_{ljk} \nabla_m v, \tag{3.2}$$

$$\nabla_{ijkl} v - \nabla_{jikl} v = R^m_{kij} \nabla_{ml} v + R^m_{lij} \nabla_{km} v. \tag{3.3}$$

From (3.2) and (3.3) we obtain

$$\begin{aligned} \nabla_{ijkl} v - \nabla_{klij} v = {} & R^m_{ljk} \nabla_{im} v + \nabla_i R^m_{ljk} \nabla_m v + R^m_{lik} \nabla_{jm} v \\ & + R^m_{jik} \nabla_{lm} v + R^m_{jil} \nabla_{km} v + \nabla_k R^m_{jil} \nabla_m v. \end{aligned} \tag{3.4}$$

Let $u \in C^4(M)$ be an admissible solution of equation (1.1). Under orthonormal local frames $e_1, \ldots, e_n$, equation (1.1) is expressed in the form

$$F(U_{ij}) := f(\lambda[U_{ij}]) = \psi \tag{3.5}$$

where $U_{ij} = \nabla_{ij} u + \chi_{ij}$. For simplicity, we shall still write equation (1.1) in the form (3.5) even if $e_1, \ldots, e_n$ are not necessarily orthonormal, although more precisely it should be

$$F(\gamma^{ik} U_{kl} \gamma^{lj}) = \psi$$

where $\{\gamma^{ij}\}$ is the square root of $\{g^{ij}\}$: $\gamma^{ik}\gamma^{kj} = g^{ij}$; as long as we use covariant derivatives whenever we differentiate the equation it will make no difference.

We now begin the proof of (1.8). Let

$$W = \max_{x\in\bar{M}} \max_{\xi\in T_xM^n, |\xi|=1} (\nabla_{\xi\xi}u + \chi(\xi,\xi))e^\eta$$

where η is a function to be determined. Suppose $W > 0$ and is achieved at an interior point $x_0 \in M$ for some unit vector $\xi \in T_{x_0}M^n$. Choose smooth orthonormal local frames $e_1, \ldots, e_n$ about x_0 such that $e_1(x_0) = \xi$ and $\{U_{ij}(x_0)\}$ is diagonal. We may also assume that $\nabla_i e_j = 0$ and therefore $\Gamma_{ij}^k = 0$ at x_0 for all $1 \leq i,j,k \leq n$. At the point x_0 where the function $\log U_{11} + \eta$ (defined near x_0) attains its maximum, we have for $i = 1, \ldots, n$,

$$\frac{\nabla_i U_{11}}{U_{11}} + \nabla_i \eta = 0, \tag{3.6}$$

$$\frac{\nabla_{ii} U_{11}}{U_{11}} - \Big(\frac{\nabla_i U_{11}}{U_{11}}\Big)^2 + \nabla_{ii}\eta \leq 0. \tag{3.7}$$

Here we wish to add some explanations which might be helpful to the reader. First we note that $U_{1j}(x_0) = 0$ for $j \geq 2$ so $\{U_{ij}(x_0)\}$ can be diagonalized. To see this let $e^\theta = e_1 \cos\theta + e_j \sin\theta$. Then

$$U_{e^\theta e^\theta}(x_0) = U_{11}\cos^2\theta + 2U_{1j}\sin\theta\cos\theta + U_{jj}\sin^2\theta$$

has a maximum at $\theta = 0$. Therefore,

$$\frac{d}{d\theta} U_{e^\theta e^\theta}(x_0)\Big|_{\theta=0} = 0.$$

This gives $U_{1j}(x_0) = 0$.

Next, at x_0 we have

$$\nabla_i(U_{11}) = \nabla_i U_{11}, \tag{3.8}$$

that is $e_i(U_{11}) = \nabla_i U_{11} \equiv \nabla^3 u(e_1, e_1, e_i) + \nabla\chi(e_1, e_1, e_i)$, and

$$\nabla_{ij}(U_{11}) = \nabla_{ij} U_{11}. \tag{3.9}$$

One can see (3.8) immediately if we assume $\Gamma_{ij}^k = 0$ at x_0 for all $1 \leq i,j,k \leq n$. In general, we have

$$\nabla_i(U_{11}) = \nabla_i U_{11} + 2\Gamma_{i1}^k U_{1k} = \nabla_i U_{11} + 2\Gamma_{i1}^1 U_{11}$$

as $U_{1k}(x_0) = 0$. On the other hand, since $e_1, \ldots e_n$ are orthonormal,

$$g(\nabla_k e_i, e_j) + g(e_i, \nabla_k e_j) = 0$$

and

$$g(\nabla_i e_1, \nabla_j e_1) + g(e_1, \nabla_i \nabla_j e_1) = 0.$$

Thus

$$\Gamma^j_{ki} + \Gamma^i_{kj} = 0 \tag{3.10}$$

and

$$\Gamma^k_{i1}\Gamma^k_{j1} + \nabla_i(\Gamma^1_{j1}) + \Gamma^k_{j1}\Gamma^1_{ik} = 0.$$

This gives $\Gamma^1_{i1} = 0$ and $\nabla_i(\Gamma^1_{j1}) = 0$. So we have (3.8).

For (3.9) we calculate directly,

$$\begin{aligned}
\nabla_{ij}(U_{11}) &= \nabla_i(\nabla_j(U_{11})) - \Gamma^k_{ij}\nabla_k(U_{11}) \\
&= \nabla_i(\nabla_j U_{11} + 2\Gamma^k_{j1}U_{1k}) - \Gamma^k_{ij}\nabla_k U_{11} \\
&= \nabla_{ij}U_{11} + \Gamma^k_{ij}\nabla_k U_{11} + 2\Gamma^k_{i1}\nabla_j U_{1k} + 2\nabla_i(\Gamma^k_{j1})U_{1k} \\
&\quad + 2\Gamma^k_{j1}\nabla_i U_{1k} + 2\Gamma^k_{j1}\Gamma^l_{i1}U_{lk} + 2\Gamma^k_{j1}\Gamma^l_{ik}U_{1l} - \Gamma^k_{ij}\nabla_k U_{11} \\
&= \nabla_{ij}U_{11} + 2\Gamma^k_{i1}\nabla_j U_{1k} + 2\Gamma^k_{j1}\nabla_i U_{1k} + 2\Gamma^k_{i1}\Gamma^k_{j1}U_{kk} - 2\Gamma^k_{i1}\Gamma^k_{j1}U_{11}
\end{aligned}$$

by (3.10) and $\nabla_i(\Gamma^1_{j1}) = 0$. Therefore we have (3.9) if $\Gamma^k_{ij} = 0$ at x_0.

We now continue our proof of (1.8). Differentiating equation (3.5) twice, we obtain at x_0,

$$F^{ij}\nabla_k U_{ij} = \nabla_k \psi, \quad \text{for all } k, \tag{3.11}$$

$$F^{ii}\nabla_{11}U_{ii} + \sum F^{ij,kl}\nabla_1 U_{ij}\nabla_1 U_{kl} = \nabla_{11}\psi. \tag{3.12}$$

Here and throughout rest of the paper, $F^{ij} = F^{ij}(\{U_{ij}\})$. By (3.4),

$$\begin{aligned}
F^{ii}\nabla_{ii}U_{11} &\geq F^{ii}\nabla_{11}U_{ii} + 2F^{ii}R_{1i1i}(\nabla_{11}u - \nabla_{ii}u) - C\sum F^{ii} \\
&\geq F^{ii}\nabla_{11}U_{ii} - C(1+U_{11})\sum F^{ii}.
\end{aligned} \tag{3.13}$$

Here we note that C depends on the gradient bound $|\nabla u|_{C^0(\bar{M})}$. From (3.7), (3.12) and (3.13) we derive

$$U_{11}F^{ii}\nabla_{ii}\eta \leq E - \nabla_{11}\psi + C(1+U_{11})\sum F^{ii} \tag{3.14}$$

where

$$E \equiv F^{ij,kl}\nabla_1 U_{ij}\nabla_1 U_{kl} + \frac{1}{U_{11}}F^{ii}(\nabla_i U_{11})^2.$$

To estimate E we follow the idea of Urbas [52]. Let $0 < s < 1$ (to be chosen) and

$$J = \{i : U_{ii} \leq -sU_{11}\}, \quad K = \{i > 1 : U_{ii} > -sU_{11}\}.$$

It was shown by Andrews [1] and Gerhardt [13] (see also [52]) that

$$-F^{ij,kl}\nabla_1 U_{ij}\nabla_1 U_{kl} \geq \sum_{i\neq j}\frac{F^{ii}-F^{jj}}{U_{jj}-U_{ii}}(\nabla_1 U_{ij})^2.$$

Therefore,

$$\begin{aligned}
-F^{ij,kl}\nabla_1 U_{ij}\nabla_1 U_{kl} &\geq 2\sum_{i\geq 2}\frac{F^{ii}-F^{11}}{U_{11}-U_{ii}}(\nabla_1 U_{i1})^2 \\
&\geq 2\sum_{i\in K}\frac{F^{ii}-F^{11}}{U_{11}-U_{ii}}(\nabla_1 U_{i1})^2 \\
&\geq \frac{2}{(1+s)U_{11}}\sum_{i\in K}(F^{ii}-F^{11})(\nabla_1 U_{i1})^2 \\
&\geq \frac{2(1-s)}{(1+s)U_{11}}\sum_{i\in K}(F^{ii}-F^{11})[(\nabla_i U_{11})^2 - C/s].
\end{aligned} \tag{3.15}$$

We now fix $s \leq 1/3$ and hence

$$\frac{2(1-s)}{1+s} \geq 1.$$

From (3.15) and (3.6) it follows that

$$\begin{aligned}
E &\leq \frac{1}{U_{11}}\sum_{i\in J}F^{ii}(\nabla_i U_{11})^2 + \frac{C}{U_{11}}\sum_{i\in K}F^{ii} + \frac{CF^{11}}{U_{11}}\sum_{i\notin J}(\nabla_i U_{11})^2 \\
&\leq U_{11}\sum_{i\in J}F^{ii}(\nabla_i\eta)^2 + \frac{C}{U_{11}}\sum F^{ii} + CU_{11}F^{11}\sum_{i\notin J}(\nabla_i\eta)^2.
\end{aligned} \tag{3.16}$$

Let

$$\eta = \phi(|\nabla u|^2) + a(\underline{u} - u)$$

where ϕ is a positive function, $\phi' > 0$, and a is a positive constant. We calculate

$$\begin{aligned}
\nabla_i\eta &= 2\phi'\nabla_k u\nabla_{ik}u + a\nabla_i(\underline{u}-u) \\
&= 2\phi'(U_{ii}\nabla_i u - \chi_{ik}\nabla_k u) + a\nabla_i(\underline{u}-u),
\end{aligned}$$

$$\nabla_{ii}\eta = 2\phi'(\nabla_{ik}u\nabla_{ik}u + \nabla_k u\nabla_{iik}u) + 2\phi''(\nabla_k u\nabla_{ik}u)^2 + a\nabla_{ii}(\underline{u}-u).$$

Therefore,

$$\sum_{i\in J}F^{ii}(\nabla_i\eta)^2 \leq 8(\phi')^2\sum_{i\in J}F^{ii}(\nabla_k u\nabla_{ik}u)^2 + Ca^2\sum_{i\in J}F^{ii}, \tag{3.17}$$

$$\sum_{i\notin J}(\nabla_i\eta)^2 \leq C(\phi')^2 U_{11}^2 + C(\phi')^2 + Ca^2 \tag{3.18}$$

and by (3.11),

$$\begin{aligned} F^{ii}\nabla_{ii}\eta \geq\, & \phi' F^{ii}U_{ii}^2 + 2\phi'' F^{ii}(\nabla_k u \nabla_{ik} u)^2 \\ & + aF^{ii}\nabla_{ii}(\underline{u} - u) - C\phi'\Big(1 + \sum F^{ii}\Big). \end{aligned} \tag{3.19}$$

Let $\phi(t) = b(1+t)^2$; we may assume $\phi'' - 4(\phi')^2 = 2b(1 - 8\phi) \geq 0$ in any fixed interval $[0, C_1]$ by requiring $b > 0$ sufficiently small. Combining (3.14), (3.16), (3.17), (3.18) and (3.19), we obtain

$$\begin{aligned} \phi' F^{ii}U_{ii}^2 + aF^{ii}\nabla_{ii}(\underline{u} - u) \leq\, & Ca^2 \sum_{i\in J} F^{ii} + C((\phi')^2 U_{11}^2 + A^2)F^{11} \\ & - \frac{\nabla_{11}\psi}{U_{11}} + C\Big(1 + \sum F^{ii}\Big). \end{aligned} \tag{3.20}$$

Suppose $U_{11}(x_0) > R$ sufficiently large and apply Theorem 2.5 to $A = \{\nabla_{ij}\underline{u} + \chi_{ij}\}$ and $B = \{U_{ij}\}$ at x_0. We see that

$$F^{ii}\nabla_{ii}(\underline{u} - u) = F^{ii}[(\nabla_{ii}\underline{u} + \chi_{ii}) - U_{ii}] \geq \theta\Big(1 + \sum F^{ii}\Big).$$

Plug this into (3.20) and fix a sufficiently large; since $|\nabla_{11}\psi| \leq CU_{11}$ if $\psi = \psi(x, u)$ we derive

$$\phi' F^{ii}U_{ii}^2 \leq Ca^2 \sum_{i\in J} F^{ii} + C((\phi')^2 U_{11}^2 + a^2)F^{11}. \tag{3.21}$$

Note that

$$F^{ii}U_{ii}^2 \geq F^{11}U_{11}^2 + \sum_{i\in J} F^{ii}U_{ii}^2 \geq F^{11}U_{11}^2 + s^2 U_{11}^2 \sum_{i\in J} F^{ii}. \tag{3.22}$$

Fixing b sufficiently small we obtain from (3.21) a bound $U_{11} \leq Ca/\sqrt{b}$. This implies (1.8), and (1.9) when M is closed.

4. Boundary estimates

In this section we establish the boundary estimate (1.13) under the assumptions of Theorem 1.5. Throughout this section we assume the function $\varphi \in C^4(\partial M)$ is extended to a C^4 function on $\bar{M}$, still denoted φ.

For a point x_0 on ∂M, we shall choose smooth orthonormal local frames $e_1, \ldots, e_n$ around x_0 such that when restricted to ∂M, e_n is normal to ∂M.

Let $\rho(x)$ denote the distance from x to x_0,

$$\rho(x) \equiv \mathrm{dist}_{M^n}(x, x_0),$$

and $M_\delta = \{x \in M : \rho(x) < \delta\}$. Since ∂M is smooth we may assume the distance function to ∂M

$$d(x) \equiv \mathrm{dist}(x, \partial M)$$

is smooth in M_{δ_0} for fixed $\delta_0 > 0$ sufficiently small (depending only on the curvature of M and the principal curvatures of ∂M.) Since $\nabla_{ij}\rho^2(x_0) = 2\delta_{ij}$, we may assume ρ is smooth in M_{δ_0} and

$$\{\delta_{ij}\} \leq \{\nabla_{ij}\rho^2\} \leq 3\{\delta_{ij}\} \quad \text{in} \quad M_{\delta_0}. \tag{4.1}$$

The following lemma which crucially depends on Theorem 2.5 plays key roles in our boundary estimates.

Lemma 4.1. *There exist some uniform positive constants t, δ, ε sufficiently small and N sufficiently large such that the function*

$$v = (u - \underline{u}) + td - \frac{Nd^2}{2} \tag{4.2}$$

satisfies $v \geq 0$ on $\bar{M}_\delta$ and

$$F^{ij}\nabla_{ij}v \leq -\varepsilon\Big(1 + \sum F^{ii}\Big) \ \ \textit{in } M_\delta. \tag{4.3}$$

Proof. We note that to ensure $v \geq 0$ in $\bar{M}_\delta$ we may require $\delta \leq 2t/N$ after t, N being fixed. Obviously,

$$\begin{aligned} F^{ij}\nabla_{ij}v &= F^{ij}\nabla_{ij}(u - \underline{u}) + (t - Nd)F^{ij}\nabla_{ij}d - NF^{ij}\nabla_i d\nabla_j d \\ &\leq C_1(t + Nd)\sum F^{ii} + F^{ij}\nabla_{ij}(u - \underline{u}) - NF^{ij}\nabla_i d\nabla_j d. \end{aligned} \tag{4.4}$$

Fix $\varepsilon > 0$ sufficiently small and $R \geq R_A$ so that Theorem 2.5 holds for $A = \{\nabla_{ij}\underline{u} + \chi_{ij}\}$ and $B = \{U_{ij}\}$ at every point in $\bar{M}_{\delta_0}$. Let $\lambda = \lambda[\{U_{ij}\}]$ be the eigenvalues of $\{U_{ij}\}$. At a fixed point in M_δ we consider two cases: (a) $|\lambda| \leq R$; and (b) $|\lambda| > R$.

In case (a) there are uniform bounds (depending on R)

$$0 < c_1 \leq \{F^{ij}\} \leq C_1$$

and therefore $F^{ij}\nabla_i d\nabla_j d \geq c_1$ since $|\nabla d| \equiv 1$. We may fix N large enough so that (4.3) holds for any $t, \varepsilon \in (0, 1]$, as long as δ is sufficiently small.

In case (b) by Theorem 2.5 and (4.4) we may further require t and δ so that (4.3) holds for some different (smaller) $\varepsilon > 0$. $\square$

We now start the proof of (1.13). Consider a point $x_0 \in \partial M$. Since $u - \underline{u} = 0$ on ∂M we have

$$\nabla_{\alpha\beta}(u-\underline{u}) = -\nabla_n(u-\underline{u})\mathit{\Pi}(e_\alpha, e_\beta), \;\; \forall\, 1 \leq \alpha, \beta < n \;\text{ on } \partial M \tag{4.5}$$

where $\mathit{\Pi}$ denotes the second fundamental form of ∂M. Therefore,

$$|\nabla_{\alpha\beta} u| \leq C, \;\; \forall\, 1 \leq \alpha, \beta < n \;\text{ on }\; \partial M. \tag{4.6}$$

To estimate the mixed tangential-normal and pure normal second derivatives we note the following formula

$$\nabla_{ij}(\nabla_k u) = \nabla_{ijk} u + \Gamma^l_{ik}\nabla_{jl} u + \Gamma^l_{jk}\nabla_{il} u + \nabla_{\nabla_{ij}e_k} u.$$

By (3.11), therefore,

$$\begin{aligned} |F^{ij}\nabla_{ij}\nabla_k(u-\varphi)| \leq & \, 2F^{ij}\Gamma^l_{ik}\nabla_{jl} u + C\Big(1 + \sum F^{ii}\Big) \\ \leq & \, C\Big(1 + \sum f_i|\lambda_i| + \sum f_i\Big). \end{aligned} \tag{4.7}$$

Let

$$\mathit{\Psi} = A_1 v + A_2\rho^2 - A_3 \sum_{\beta<n} |\nabla_\beta(u-\varphi)|^2. \tag{4.8}$$

By (4.7) we have

$$\begin{aligned} F^{ij}\nabla_{ij}|\nabla_\beta(u-\varphi)|^2 = & \, 2F^{ij}\nabla_\beta(u-\varphi)\nabla_{ij}\nabla_\beta(u-\varphi) \\ & + 2F^{ij}\nabla_i\nabla_\beta(u-\varphi)\nabla_j\nabla_\beta(u-\varphi) \\ \geq & \, F^{ij}U_{i\beta}U_{j\beta} - C\Big(1 + \sum f_i|\lambda_i| + \sum f_i\Big). \end{aligned} \tag{4.9}$$

For fixed $1 \leq \alpha < n$, by Lemma 4.1, Proposition 2.6 and Corollary 2.8 we see that

$$F^{ij}\nabla_{ij}(\mathit{\Psi} \pm \nabla_\alpha(u-\varphi)) \leq 0, \;\; \forall \;\; \text{in } M_\delta \tag{4.10}$$

and $\mathit{\Psi} \pm \nabla_\alpha(u-\varphi) \geq 0$ on ∂M_δ when $A \gg A_2 \gg A_3 \gg 1$. By the maximum principle we derive $\mathit{\Psi} \pm \nabla_\alpha(u-\varphi) \geq 0$ in M_δ and therefore

$$|\nabla_{n\alpha} u(x_0)| \leq \nabla_n \mathit{\Psi}(x_0) \leq C, \;\; \forall\, \alpha < n. \tag{4.11}$$

It remains to derive

$$\nabla_{nn} u(x_0) \leq C. \tag{4.12}$$

Following an idea of Trudinger [50] we show that there are uniform constants c_0, R_0 such that for all $R > R_0$, $(\lambda'[\{U_{\alpha\beta}(x_0)\}], R) \in \Gamma$ and

$$f(\lambda'[\{U_{\alpha\beta}(x_0)\}], R) \geq \psi(x_0) + c_0$$

where $\lambda'[\{U_{\alpha\beta}\}] = (\lambda'_1, \cdots, \lambda'_{n-1})$ denotes the eigenvalues of the $(n-1) \times (n-1)$ matrix $\{U_{\alpha\beta}\}$ $(1 \leq \alpha, \beta \leq n-1)$. Suppose we have found such c_0 and R_0. By Lemma 1.2 of [5], from estimates (4.6) and (4.11) we can find $R_1 \geq R_0$ such that if $U_{nn}(x_0) > R_1$,

$$f(\lambda[\{U_{ij}(x_0)\}]) \geq f(\lambda'[\{U_{\alpha\beta}(x_0)\}], U_{nn}(x_0)) - \frac{c_0}{2}.$$

By equation (1.1) this gives a desired bound $U_{nn}(x_0) \leq R_1$ for otherwise, we would have

$$f(\lambda[\{U_{ij}(x_0)\}]) \geq \psi(x_0) + \frac{c_0}{2}.$$

For $R > 0$ and a symmetric $(n-1)^2$ matrix $\{r_{\alpha\beta}\}$ with $(\lambda'[\{r_{\alpha\beta}(x_0)\}], R) \in \Gamma$, define

$$\tilde{F}[r_{\alpha\beta}] \equiv f(\lambda'[\{r_{\alpha\beta}\}], R)$$

and consider

$$m_R \equiv \min_{x_0 \in \partial M} \tilde{F}[U_{\alpha\beta}(x_0)] - \psi(x_0).$$

Note that $\tilde{F}$ is concave and m_R is increasing in R by (1.3), and that

$$c_R \equiv \inf_{\partial M}(\tilde{F}[\underline{U}_{\alpha\beta}] - F[\underline{U}_{ij}]) \geq \inf_{\partial M}(\tilde{F}[\underline{U}_{\alpha\beta}] - f(\lambda'[\underline{U}_{\alpha\beta}], \underline{U}_{nn})) > 0$$

when R is sufficiently large.

We wish to show $m_R > 0$ for R sufficiently large. Suppose m_R is achieved at a point $x_0 \in \partial M$. Choose local orthonormal frames around x_0 as before and let

$$\tilde{F}_0^{\alpha\beta} = \frac{\partial \tilde{F}}{\partial r_{\alpha\beta}}[U_{\alpha\beta}(x_0)].$$

Since $\tilde{F}$ is concave, for any symmetric matrix $\{r_{\alpha\beta}\}$ with $(\lambda'[\{r_{\alpha\beta}\}], R) \in \Gamma$,

$$\tilde{F}_0^{\alpha\beta}(r_{\alpha\beta} - U_{\alpha\beta}(x_0)) \geq \tilde{F}[r_{\alpha\beta}] - \tilde{F}[U_{\alpha\beta}(x_0)]. \tag{4.13}$$

In particular,

$$\tilde{F}_0^{\alpha\beta}U_{\alpha\beta} - \psi - \tilde{F}_0^{\alpha\beta}U_{\alpha\beta}(x_0) + \psi(x_0) \geq \tilde{F}[U_{\alpha\beta}] - \psi - m_0 \geq 0 \text{ on } \partial M. \tag{4.14}$$

By (4.5) we have on ∂M,

$$U_{\alpha\beta} = \underline{U}_{\alpha\beta} - \nabla_n(u - \underline{u})\sigma_{\alpha\beta} \tag{4.15}$$

where $\sigma_{\alpha\beta} = \langle \nabla_\alpha e_\beta, e_n \rangle$; note that $\sigma_{\alpha\beta} = \mathit{\Pi}(e_\alpha, e_\beta)$ on ∂M. It follows that

$$\begin{aligned}\nabla_n(u - \underline{u})\tilde{F}_0^{\alpha\beta}\sigma_{\alpha\beta}(x_0) &= \tilde{F}_0^{\alpha\beta}(\underline{U}_{\alpha\beta}(x_0) - U_{\alpha\beta}(x_0))\\ &\geq \tilde{F}[\underline{U}_{\alpha\beta}(x_0)] - \tilde{F}[U_{\alpha\beta}(x_0)]\\ &= \tilde{F}[\underline{U}_{\alpha\beta}(x_0)] - \psi(x_0) - m_R \geq c_R - m_R.\end{aligned}$$

Consequently, if

$$\nabla_n(u - \underline{u})(x_0)\tilde{F}_0^{\alpha\beta}\sigma_{\alpha\beta}(x_0) \leq c_R/2$$

then $m_R \geq c_R/2$ and we are done.

Suppose now that

$$\nabla_n(u - \underline{u})(x_0)\tilde{F}_0^{\alpha\beta}\sigma_{\alpha\beta}(x_0) > \frac{c_R}{2}$$

and let $\eta \equiv \tilde{F}_0^{\alpha\beta}\sigma_{\alpha\beta}$. Note that

$$\eta(x_0) \geq c_R/2\nabla_n(u - \underline{u})(x_0) \geq 2\epsilon_1 c_R \tag{4.16}$$

for some uniform $\epsilon_1 > 0$ independent of R. We may assume $\eta \geq \epsilon_1 c_R$ on $\bar{M}_\delta$ by requiring δ small. Define in M_δ,

$$\begin{aligned}\mathit{\Phi} =& -\nabla_n(u - \varphi) + \frac{1}{\eta}\tilde{F}_0^{\alpha\beta}(\nabla_{\alpha\beta}\varphi + \chi_{\alpha\beta} - U_{\alpha\beta}(x_0)) - \frac{\psi - \psi(x_0)}{\eta}\\ \equiv& -\nabla_n(u - \varphi) + Q.\end{aligned}$$

We have $\mathit{\Phi}(0) = 0$ and $\mathit{\Phi} \geq 0$ on ∂M near 0 by (4.14) since

$$\nabla_{\alpha\beta}u = \nabla_{\alpha\beta}\varphi - \nabla_n(u - \varphi)\sigma_{\alpha\beta} \text{ on } \partial M,$$

while by (4.7),

$$F^{ij}\nabla_{ij}\mathit{\Phi} \leq -F^{ij}\nabla_{ij}\nabla_n u + C\sum F^{ii} \leq C\Big(1 + \sum f_i|\lambda_i| + \sum f_i\Big). \tag{4.17}$$

Consider the function $\mathit{\Psi}$ defined in (4.8). Applying Lemma 4.1, Proposition 2.6 and Corollary 2.8 as before for $A_1 \gg A_2 \gg A_3 \gg 1$ we derive $\mathit{\Psi} + \mathit{\Phi} \geq 0$ on ∂M_δ and

$$F^{ij}\nabla_{ij}(\mathit{\Psi} + \mathit{\Phi}) \leq 0 \text{ in } M_\delta. \tag{4.18}$$

By the maximum principle, $\mathit{\Psi} + \mathit{\Phi} \geq 0$ in M_δ. Thus $\mathit{\Phi}_n(x_0) \geq -\nabla_n\mathit{\Psi}(x_0) \geq -C$. This gives $\nabla_{nn}u(x_0) \leq C$.

So we have an *a priori* upper bound for all eigenvalues of $\{U_{ij}(x_0)\}$. Consequently, $\lambda[\{U_{ij}(x_0)\}]$ is contained in a compact subset of Γ by (1.5), and therefore

$$m_R = \tilde{F}[U_{\alpha\beta}(x_0)] - \psi(x_0) > 0$$

when R is sufficiently large. This completes the proof of (1.13).

5. Further results and remarks

5.1. **The results of Li** [43] **and Urbas** [52]**.** In [43] Li treated equation (1.1) with $\chi = g$ on closed manifolds with nonnegative sectional curvature, and in various other situations. His basic assumptions used in the second derivative estimates include (1.3), (1.4), (1.6) as well as the following:

$$L_0 := \lim_{\lambda\to 0, \lambda\in\Gamma} \inf f(\lambda) > -\infty, \tag{5.1}$$

and

$$\lim_{|\lambda|\to+\infty, \lambda\in\partial\Gamma^\sigma} \sum f_i(\lambda) = +\infty, \ \ \forall\, \sigma > \sup_{\partial\Gamma} f. \tag{5.2}$$

Li also derived the gradient estimates under the same assumptions.

Urbas [52] was able remove the nonnegative curvature condition in [43], and showed that assumption (5.2) could be replaced by

$$\sum f_i(\lambda) \geq \delta_\sigma, \ \ \forall\, \lambda \in \partial\Gamma^\sigma, \ \sigma > \sup_{\partial\Gamma} f, \tag{5.3}$$

and

$$\lim_{|\lambda|\to+\infty, \lambda\in\partial\Gamma^\sigma} \sum f_i(\lambda)\lambda_i^2 = +\infty, \ \ \forall\, \sigma > \sup_{\partial\Gamma} f. \tag{5.4}$$

The main assumption in [52] for the gradient estimates is (1.15) which was also used in earlier papers for gradient estimates [38], [44], [49], [22], [10].

The following lemma clarifies relations between assumptions (5.1), (5.2) and (1.7).

Lemma 5.1. *Suppose f satisfies* (1.3), (1.4), (5.1) *and* (5.2)*. Then $\Gamma_n^+ \subset \mathcal{C}_\sigma^+$ for any $\sigma > \sup_{\partial\Gamma} f$. Consequently, condition* (1.7) *is satisfied if $\chi > 0$.*

Proof. Let $\lambda \in \Gamma$. By the concavity of f,

$$\sum f_{\lambda_i}(\lambda)(\delta - \lambda_i) \geq f(\delta\mathbf{1}) - f(\lambda)$$

for any $\delta > 0$. Letting δ tend to 0, we obtain by (5.1),

$$\sum f_{\lambda_i}(\lambda)\lambda_i \leq f(\lambda) - L_0. \tag{5.5}$$

Let $\mu \in \Gamma_n^+$ and assume $\mu_1 \geq \cdots \geq \mu_n > 0$. Then for $\lambda \in \Gamma^\sigma$

$$\sum f_{\lambda_i}(\lambda)(\mu_i - \lambda_i) \geq \mu_n \sum f_{\lambda_i}(\lambda) - \sum f_\lambda(\lambda)\lambda_i \geq \mu_n \sum f_{\lambda_i}(\lambda) + L_0 - \sigma > 0$$

by (5.2) when $|\lambda|$ is sufficiently large. This clearly implies $\mu \in \mathcal{C}_\sigma^+$. □

Concerning condition (5.4) we have the following observation.

Proposition 5.2. *Theorem 1.5 still holds with assumption* (1.12) *replaced by* (5.4)*, and therefore so does Theorem 1.7.*

Proof. In the function Ψ defined in (4.8) we replace v by $(u - \underline{u})$ and call this new function $\tilde{\Psi}$. Since $\underline{u}$ is an admissible subsolution, by the concavity of f there exists $\epsilon > 0$ such that

$$F^{ij}\nabla_{ij}(\underline{u} - u) \geq \epsilon \sum F^{ii} - C.$$

Applying Proposition 2.6 and Corollary 2.8, by assumption (5.4) we may choose $A_1 \gg A_2 \gg A_3 \gg 1$ as before such that

$$F^{ij}\nabla_{ij}\tilde{\Psi} \leq -C\Big(1 + \sum f_i(1 + \lambda_i^2)\Big)$$

for any $C > 0$ when $|\lambda|$ is sufficiently large. The rest of the proof is now same as that of Theorem 1.5. $\square$

5.2. **The gradient estimates.** Building upon the estimates in Theorems 1.1 and 1.5 with the aid of Evans-Krylov theorem, one needs to derive *a prior* C^1 estimates in order to establish existence of solutions to equation (1.1) either on closed manifolds or for the Dirichlet problem on manifolds with boundary, using standard analytic tools such as the continuity methods and degree arguments. It seems an interesting question whether one can prove gradient estimates under assumption (1.7). We wish to come back to the problem in future work. Here we only list some results that were more or less already known to Li [43] and Urbas [52].

Proposition 5.3. *Let $u \in C^3(\bar{M})$ be an admissible solution of equation* (1.1) *where $\psi \in C^1(\bar{M})$. Suppose f satisfies* (1.3)-(1.5)*. Then*

$$\max_{\bar{M}} |\nabla u| \leq C\big(1 + \max_{\partial M} |\nabla u|\big) \tag{5.6}$$

where C depends on $|u|_{C^0(\bar{M})}$, under any of the following additional assumptions: **(i)** $\Gamma = \Gamma_n^+$*;* **(ii′)** (1.7)*, $\psi_u \geq 0$ and that (M, g) has nonnegative sectional curvature;* **(iii′)** (1.10) *and* (1.15) *for $|\lambda|$ sufficiently large.*

Proof. Consider case **(i)**: $\Gamma = \Gamma_n^+$. For fixed $A > 0$ suppose $Au + |\nabla u|^2$ has a maximum at an interior point $x_0 \in M$. Then $A\nabla_i u + 2\nabla_k u \nabla_{ki} u = \nabla_k u(A\delta_{ki} + \nabla_{ki} u) = 0$ at x_0 for all $1 \leq i \leq n$. This implies $\nabla u(x_0) = 0$ when A is sufficiently large. Therefore,

$$\sup_M |\nabla u|^2 \leq A\Big(\sup_{\partial M} u - \inf_M u\Big) + \sup_{\partial M} |\nabla u|^2.$$

Case **(iii′)** was proved by Urbas [52] under the additional assumption (5.3) which is implied by (1.10). Indeed, by the concavity of f and (1.10),

$$A \sum f_{\lambda_i}(\lambda) \geq \sum f_{\lambda_i}(\lambda)\lambda_i + f(A\mathbf{1}) - f(\lambda) \geq f(A\mathbf{1}) - \sigma$$

for any $\lambda \in \Gamma$, $f(\lambda) = \sigma$. Fixing A sufficiently large gives (5.3).

Case **(ii′)**. Gradient estimates were established by Li [43] on closed manifolds with nonnegative sectional curvature under the additional assumptions (5.1) and (5.2). His proof can be modified to replace (5.1) and (5.2) by (1.7). We only outline the proof.

Suppose $|\nabla u|^2 e^{\phi}$ achieves a maximum at an interior point $x_0 \in M$. Then at x_0,

$$\frac{2\nabla_k u \nabla_{ik} u}{|\nabla u|^2} + \nabla_i \phi = 0,$$

$$2F^{ij}(\nabla_k u \nabla_{jik} + \nabla_{ik} u \nabla_{jk} u) + |\nabla u|^2 F^{ij}(\nabla_{ij}\phi - \nabla_i \phi \nabla_j \phi) \leq 0.$$

Following [43] we use the nonnegative sectional curvature condition to derive

$$|\nabla u|^2 F^{ij}(\nabla_{ij}\phi - \nabla_i \phi \nabla_j \phi) \leq C|\nabla u| - \psi_u |\nabla u|^2. \tag{5.7}$$

Now let $\phi = A(\underline{u} - u)^2$ and fix $A > 0$ sufficiently small. By (1.7) and Theorem 2.5 we derive a bound $|\nabla u(x_0)| \leq C$ if $|\lambda[\nabla^2 u + \chi](x_0)| \geq R$ for R sufficiently large.

Suppose $|\lambda[\nabla^2 u + \chi](x_0)| \leq R$. Then by (1.3) and (1.5), there exists $C_1 > 0$ depending on R such that at x_0,

$$\frac{g^{-1}}{C_1} \leq \{F^{ij}\} \leq C_1 g^{-1}.$$

From (5.7),

$$\begin{aligned} \frac{C}{|\nabla u|} &\geq 2AF^{ij}\nabla_{ij}(\underline{u} - u) + 2A(1-2A)F^{ij}\nabla_i(\underline{u} - u)\nabla_j(\underline{u} - u) \\ &\geq 2A(1-2A)C_1^{-1}|\nabla(\underline{u} - u)|^2 - CA. \end{aligned}$$

We derive a bound for $|\nabla u(x_0)|$ again. □

5.3. **An example.** Consider the function

$$P_k(\lambda) := \prod_{i_1 < \cdots < i_k} (\lambda_{i_1} + \cdots + \lambda_{i_k}), \quad 1 \leq k \leq n$$

defined in the cone

$$\mathcal{P}_k := \{\lambda \in \mathbb{R}^n : \lambda_{i_1} + \cdots + \lambda_{i_k} > 0\}.$$

Obviously,

$$\sup_{\partial \mathcal{P}_k} P_k = 0.$$

Let $f = \log P_k$. Then

$$\frac{\partial f}{\partial \lambda_i} = \sum_{i_2<\cdots<i_k; i_l\neq i} \frac{1}{\lambda_i + \lambda_{i_2} + \cdots + \lambda_{i_k}},$$

$$\frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j} = -\sum_{i_3<\cdots<i_k; i_l\neq i,j} \frac{1}{(\lambda_i + \lambda_j + \lambda_{i_3} + \cdots + \lambda_{i_k})^2}.$$

Therefore $f = \log P_2$ satisfies (1.3) and (1.4) in $\mathcal{P}_2$. Moreover, $\Gamma^\sigma \equiv \{P_2 > \sigma\}$ is strictly convex and $\mathcal{C}_\sigma^+ = \mathcal{P}_2$. Consequently, Corollary 1.8 holds for $f = P_2$.

In [36] Huisken and Sinestrari studied the mean curvature flow of hypersurfaces with principal curvatures $(\kappa_1, \ldots, \kappa_n) \in \mathcal{P}_2$; they call such hypersurfaces *two-convex*.

There seem interesting cases among the quotients P_k/P_l but the situation is more complicated. We hope to discuss them in future work. Note that $P_1 = \sigma_n$, $P_n = \sigma_1$.

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